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Collective field description of matrix cosmologies

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Abstract: We study the Das-Jevicki collective field description of arbitrary classical solutions in the $c = 1$ matrix model, which are believed to describe nontrivial spacetime backgrounds in 2d string theory. Our analysis naturally includes the case of a Fermi droplet cosmology: a finite size droplet of Fermi fluid, made up of a finite number of eigenvalues. We analyze properties of the coordinates in which the metric in the collective field theory is trivial, and comment on the form of the interaction terms in these coordinates.

Keywords: Tachyon Condensation, Matrix Models.

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1. Introduction

The soluble string theory in 1+1 dimensions is a rich toy model for the study of nonperturbative phenomena often not accessible to analysis in higher dimensional theories. Of such phenomena, one class are the processes with a nontrivial time evolution. Examples include tachyon condensation, creation and evaporation of black holes, and cosmological evolution.

An important step towards the study of time-dependent phenomena in the $c = 1$ matrix model for the 2d string was the description of D0-brane decay, or open string tachyon condensation [1, 2]. The matrix model provides an exact picture of the time evolution as the classical motion of a single matrix eigenvalue; its predictions were compared to worldsheet string analysis and found to agree.

Classical collective motions of the entire Fermi sea, as opposed to a motion of a single eigenvalue, were described for example in [3, 4]. These describe nontrivial time dependent backgrounds for the 2d string theory and were interpreted as closed string tachyon condensation in [5]. Another class of time-dependent solutions — droplets of large but finite number of eigenvalues, corresponding to closed universe cosmologies — was proposed in [5]. Since these classical time-dependent solutions of the matrix model correspond to large motions of the Fermi surface, small fluctuations about the Fermi surface carry important information about propagation of stringy spacetime fields. As we will review below, these small fluctuations can be described in the Das-Jevicki collective field approach by a 2d effective field theory, whose action generically contains a nontrivial, time-varying metric.

A step toward understanding these time-dependent solutions was taken by Alexandrov in [6], where coordinates were found in which the metric was made trivial. However, the method presented there does not extend to compact Fermi droplets. The main purpose of this note is to extend the construction of Alexandrov coordinates to arbitrary Fermi surfaces, including compact cases.

To this end, we study the Alexandrov coordinates in some detail. In section 2, we briefly review the collective field description of small fluctuations about a time-dependent Fermi surface. In section 3, we explicitly construct the Alexandrov coordinates for an arbitrary solution. In section 4, we analyze a special class of backgrounds (including some compact cases) for which the entire, interacting action is static. The collective field action for these solutions is shown to take a standard form with a time-independent coupling constant. In section 5, we construct the Alexandrov coordinates for a finite collection of fermion eigenvalues: a compact droplet cosmology. We briefly discuss the possibility of formulating a spacetime string theory interpretation of such a configuration. Finally, in the appendix, we analyze the interaction term in the effective action and show by an explicit example that it is not always possible to make it static.

2. Notation and Alexandrov coordinates

In the double scaling limit, matrix quantum mechanics is defined by the action

$$S = \frac{1}{2}\int dt\, \mathrm{Tr}\left(\dot{M}(t)^2 + M(t)^2\right), \tag{2.1}$$

where M is a hermitean matrix whose size in this limit is taken to infinity. As is well known (for reviews, see for example [7, 8]), upon quantization the singlet sector of the matrix quantum mechanics is described by an infinite number of free (noninteracting), nonrelativistic fermions representing the eigenvalues of M. The fermions inherit the same potential as the matrix M, and hence the single variable hamiltonian is

$$H = \frac{1}{2}(p^2 - x^2)\,. \tag{2.2}$$

Since the number of fermions is large, the classical limit of the theory is that of an incompressible Fermi liquid moving in phase space (x, p) under the equations of motion given by the hamiltonian (2.2). We will restrict our analysis in this note to situations where the Fermi surface (the boundary of the Fermi sea) can be given by its upper and lower branch, which we will denote with $p_\pm(x,t)$, see figure 1. It is easy to show that $p_\pm(x,t)$ satisfy

$$\partial_t p_\pm + p_\pm \partial_x p_\pm = x\,. \tag{2.3}$$

One way to directly connect the classical limit of the matrix quantum mechanics with the collective description of fermion motion is via a procedure developed by Das and Jevicki [11]. Define a field $\varphi(x,t)$ by

$$\varphi(x,t) = \frac{1}{\pi}\,\mathrm{Tr}[\delta(x - M(t))] \tag{2.4}$$

so that $\varphi(x,t)$ is the density of eigenvalues at point x and time t. In the fermion description, we have the relation

$$\varphi = \frac{p_+ - p_-}{2}\,. \tag{2.5}$$

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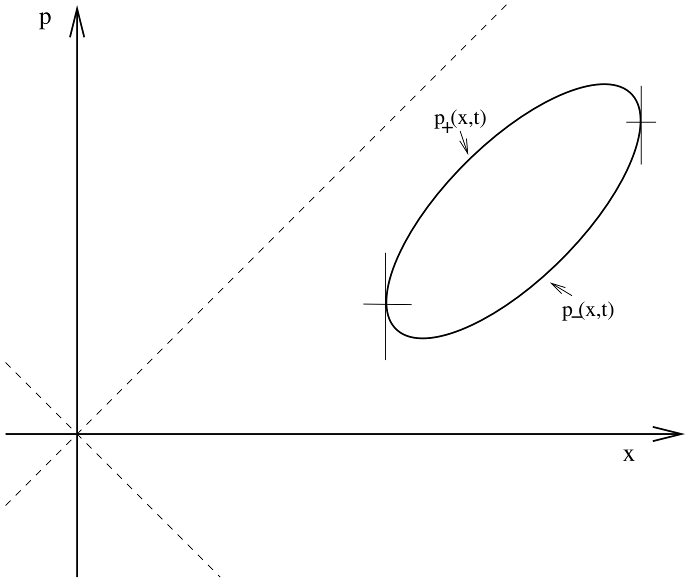

Figure 1: A compact Fermi surface in phase space. The upper and lower branches of the surface are labelled, and vertical points where they meet (and the collective theory becomes strongly coupled) are marked.

The action for the collective field is [11]

$$S = \int \frac{dt\,dx}{2\pi} \left\{ \frac{Z^2}{\varphi} - \frac{1}{3}\varphi^3 + (x^2 - 2\mu)\varphi \right\}, \tag{2.6}$$

where $Z = \int dx \partial_t \varphi$, so the equation of motion is

$$\partial_t \left(\frac{Z}{\varphi} \right) - \frac{Z}{\varphi} \partial_x \left(\frac{Z}{\varphi} \right) = \varphi \partial_x \varphi - x\,. \tag{2.7}$$

Furthermore, we have the relation [12]

$$\frac{Z}{\varphi} = -\frac{p_+ + p_-}{2}, \tag{2.8}$$

which allows us to verify that (2.7) is consistent with (2.3).

We want to consider a fixed solution $\varphi_0(x,t)$ of (2.7) and study the effective action for small fluctuations about this solution. In the string theory dual to the matrix model, this corresponds to studying the small fluctuations about a string background given by the solution $\varphi_0(x,t)$. Let $\partial_x \eta(x,t)$ denote the small fluctuations

$$\varphi = \varphi_0 + \sqrt{\pi}\partial_x \eta \tag{2.9}$$

and let $Z_0 = \int dx \partial_t \varphi_0$.

Rewriting the action and grouping terms in powers of η we find (noticing that terms linear in η vanish by the equations of motion)

$$\begin{aligned} S &= \int \frac{dt\,dx}{2\pi} \left\{ \frac{(Z_0 + \sqrt{\pi}\partial_t \eta)^2}{\varphi_0 + \sqrt{\pi}\partial_x \eta} - \frac{1}{3}(\varphi_0 + \sqrt{\pi}\partial_x \eta)^3 + (x^2 - 2\mu)(\varphi_0 + \sqrt{\pi}\,\partial_x \eta) \right\} \\ &= S_{(0)} + S_{(2)} + S_{\rm int} \end{aligned} \tag{2.10}$$

where $S_{(0)}$ has no η-dependence,

$$S_{(2)} = \frac{1}{2}\int \frac{dt\ dx}{\varphi_0}\left\{\left(\partial_t\eta - \frac{Z_0}{\varphi_0}\partial_x\eta\right)^2 - \varphi_0^2(\partial_x\eta)^2\right\}, \tag{2.11}$$

and

$$S_{\rm int} = \frac{1}{2}\int \frac{dt\ dx}{\varphi_0}\left\{-\frac{\sqrt{\pi}}{3}\varphi_0(\partial_x\eta)^3 + \left(\partial_t\eta - \frac{Z_0}{\varphi_0}\partial_x\eta\right)^2\sum_{n=1}^{\infty}(-\sqrt{\pi})^n\left(\frac{\partial_x\eta}{\varphi_0}\right)^n\right\}. \tag{2.12}$$

In [6], it is proposed that coordinates (τ, σ) exist in which $S_{(2)}$ takes a standard form of a kinetic term for a field in a flat metric

$$S_{(2)} = \int d\tau^+ d\tau^-\ \partial_{\tau^-}\eta\, \partial_{\tau^+}\eta\,, \tag{2.13}$$

where $\tau^\pm(x,t) = \tau \pm \sigma$ are the lightcone coordinates. We shall refer to the coordinates (τ, σ) as the *Alexandrov coordinates*. In [6], these coordinates were constructed from a specific form of the solution φ_0. In the next section, we prove (at least locally) their existence for all φ_0.

3. Alexandrov coordinates — existence

It is quite simple to show, using the equations of motion for the two branches of the solution (2.3), that the action (2.11) takes on the form in (2.13) as long as the coordinates $\tau^\pm$ satisfy

$$(\partial_t + p_\pm\partial_x)\tau^\pm = 0\,. \tag{3.1}$$

Equation (3.1) can easily be solved (at least locally). The exact form of the solution depends on whether the slope of the solution $p_\pm$ is steeper or shallower than 1. The regions where $\alpha(x,t) \equiv \partial_x p_\pm$ satisfies $|\alpha| > 1$ will be referred to as the steep regions, and those were $|\alpha| < 1$ will be referred to as the shallow regions. In the steep regions, we have that

$$\tau^\pm = t - \coth^{-1}(\partial_x p_\pm)\,, \tag{3.2}$$

and in the shallow regions we get

$$\tau^\pm = t - \tanh^{-1}(\partial_x p_\pm)\,. \tag{3.3}$$

The solution above is not unique — a conformal change of coordinates does not change the form of the quadratic part of the action (2.13), so any change of coordinates of the form

$$\tau'^\pm = \tau'^\pm(\tau^\pm) \tag{3.4}$$

will provide another solution to equation (3.1). For example, the following is also a good solution

$$\tau^\pm = \frac{\tanh t - \partial_x p_\pm}{1 - \partial_x p_\pm \tanh t} \tag{3.5}$$

as is

$$\tau^{\pm} = \frac{\coth t - \partial_x p_{\pm}}{1 - \partial_x p_{\pm} \coth t}\,. \tag{3.6}$$

Note that if p_+ and p_- are flat on some overlapping region ($\partial_x p_{\pm} = 0$), then these coordinates will be degenerate. However, we can easily parameterize these flat regions in a nondegenerate way so that the metric is still flat.

The solutions (3.2) and (3.3) are valid locally on steep and shallow coordinate patches respectively (modulus the degenerate case mentioned above). To create a single coordinate system, we can 'glue together' the various steep, shallow, and flat patches by using the freedom of conformal coordinate changes. While our expressions guarantee the existence of Alexandrov coordinates on each patch and while there are no obvious obstacles to the 'gluing' procedure, constructing the coordinates in this way would be very cumbersome, even in cases where the resulting coordinate systems are simple. Instead, in all the examples given in this paper, the Alexandrov coordinates are constructed by the procedure given in [6] (but see the comment at the end of section 5).

Another issue is that, as was shown in [6], the resulting coordinates often have boundaries (this will also be seen in section 5). The boundaries come in two categories. The first are timelike boundaries corresponding to the end(s) of the Fermi sea; the boundary conditions on those can be determined from the conservation of fermion number [11]. The second class of boundaries contains boundaries which are either spacelike or timelike, do not have a clear interpretation, and for which appropriate boundary conditions are not known. We will return to the issues of boundaries in the discussion in section 5.

We will close this section with a simple example as an illustration. Consider a moving hyperbolic Fermi surface given parametrically by [5]

$$\begin{aligned} x &= \sqrt{2\mu}\cosh\sigma + \lambda e^t \\ p &= \sqrt{2\mu}\sinh\sigma + \lambda e^t\,. \end{aligned} \tag{3.7}$$

In this case, we have $\varphi_0 = \sqrt{(x-\lambda e^t)^2 - 2\mu} = \sqrt{2\mu}\sinh\sigma$. The Alexandrov coordinates are given simply by σ in the parametrization above and by $\tau = t$. It is a simple matter to check that the action takes the form

$$\begin{aligned} S = \int d\tau d\sigma \Bigg\{ &\frac{1}{2}((\partial_\tau\eta)^2 - (\partial_\sigma\eta)^2) - \frac{\sqrt{\pi}}{6\varphi_0^2}(3(\partial_\tau\eta)^2(\partial_\sigma\eta) + (\partial_\sigma\eta)^3) + \\ &+ \frac{(\partial_\tau\eta)^2}{2}\sum_{n=2}^{\infty}\left(-\frac{\sqrt{\pi}(\partial_\sigma\eta)}{\varphi_0^2}\right)^n \Bigg\}\,. \end{aligned} \tag{3.8}$$

Note that the coupling diverges at the point $\sigma = 0$ which corresponds to the edge of the Fermi sea, and that it does not depend on τ.

4. Alexandrov coordinates — special case

In this section, we study a class of solutions (of which an example appeared at the end of the previous section) for which the Alexandrov coordinates can be written as

$$\sigma = \sigma(x,t)\,, \qquad \tau = \tau(t)\,. \tag{4.1}$$

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We shall see that this leads to a very restricted class of solutions, but a class which includes both infinite and finite (compact) Fermi seas. Thus, it encompasses the two generic types of dynamic solutions.

With the coordinate ansatz above, we have

$$\begin{aligned} dt\, dx &= \frac{d\tau\, d\sigma}{|\partial_x\sigma\, \partial_t\tau|} \\ \partial_x &= \partial_x\sigma\, \partial_\sigma \\ \partial_t &= \partial_t\tau\, \partial_\tau + \partial_t\sigma\, \partial_\sigma\,. \end{aligned} \tag{4.2}$$

Demanding that the kinetic term take the standard flat form

$$S_{(2)} = \int d\tau\, d\sigma \left[\frac{1}{2}\left(\partial_\tau\eta\right)^2 - \frac{1}{2}\left(\partial_\sigma\eta\right)^2\right] \tag{4.3}$$

leads to the requirements that

$$\partial_t\sigma = \frac{Z_0}{\varphi_0}\partial_x\sigma \qquad \text{and} \qquad \left|\frac{\partial_t\tau}{\partial_x\sigma}\right| = \varphi_0\,. \tag{4.4}$$

These constraints can be solved explicitly, provided that the solution is only vertical at endpoints ($\varphi_0 = 0$). Since τ depends only on t, we find that $\partial_t\tau = (\partial_\tau t)^{-1}$, $\partial_x\sigma = (\partial_\sigma x)^{-1}$, $\partial_x\tau = \partial_\sigma t = 0$,

$$\partial_\tau x = -\frac{\partial_t\sigma}{(\partial_x\sigma)(\partial_t\tau)}\,. \tag{4.5}$$

Using the first equation in (4.4) we find

$$\partial_x Z_0 = -\frac{1}{\partial_\tau t}\left[\varphi_0\partial_\tau \ln(\partial_\sigma x) + \frac{\partial_\tau x}{\partial_\sigma x}\partial_\sigma\varphi_0\right], \tag{4.6}$$

which is equal to

$$\partial_x Z_0 = \partial_t\varphi_0 = \frac{1}{\partial_\tau t}\left[\partial_\tau\varphi_0 - \frac{\partial_\tau x}{\partial_\sigma x}\partial_\sigma\varphi_0\right]. \tag{4.7}$$

Comparing these two expressions, we obtain a differential equation for φ_0

$$\partial_\tau \ln(\varphi_0) = -\partial_\tau \ln(\partial_\sigma x) \tag{4.8}$$

whose solution is clearly of the form $\varphi_0 = f(\sigma)^2(\partial_\sigma x)^{-1}$. This we can rewrite, using the second equation in (4.4), as

$$\varphi_0(\sigma,\tau) = f(\sigma)\sqrt{g(\tau)}\,, \tag{4.9}$$

where $g(\tau) = (\partial_\tau t)^{-1}$ and we assume $f(\sigma) > 0$. We also have $\partial_x\sigma = \sqrt{g}/f$.

Now we can use the equation of motion (2.7) to find the forms of f and g. Using (4.4), notice that

$$\partial_\tau = (\partial_\tau t)\partial_t + (\partial_\tau x)\partial_x = (\partial_\tau t)\left(\partial_t - \frac{Z_0}{\varphi_0}\partial_x\right), \tag{4.10}$$

so the equation of motion (2.7) implies

$$g(\tau)\partial_\sigma\partial_\tau\left(\frac{Z}{\varphi}\right) = \partial_\sigma(\varphi\partial_x\varphi) - \frac{1}{\partial_x\sigma}\,. \tag{4.11}$$

Substituting the explicit form of φ_0 in terms of f and g into this equation, we obtain

$$2g\partial_\tau^2 g - (\partial_\tau g)^2 + 4 - 4g^2\frac{\partial_\sigma^2 f}{f} = 0\,. \tag{4.12}$$

Since g only depends on τ, and f only on σ, we see that $\partial_\sigma^2 f = -\alpha f$ where α is a constant.

Consider first the situation when α is positive. Then

$$f(\sigma) = f_1 \sin(\sqrt{\alpha}(\sigma - \sigma_1))\,, \tag{4.13}$$

where f_1 and σ_1 are real numbers. To ensure that $\varphi_0 \geq 0$, we must restrict $f_1 > 0$ and $\sigma_1 \leq \sigma \leq \sigma_1 + \frac{\pi}{\sqrt{\alpha}}$. Requiring g to be real yields

$$g(\tau) = \frac{1}{\sqrt{\alpha}}\left[\sqrt{c^2+1}\,\cos(2\sqrt{\alpha}(\tau-\tau_1)) + c\right], \tag{4.14}$$

where c and τ_1 are real constants of integration. If α is negative and $|c| \geq 1$, we have

$$f(\sigma) = f_1 \sinh(\sqrt{|\alpha|}\sigma) + f_2\cosh(\sqrt{|\alpha|}\sigma) \tag{4.15}$$

and

$$g(\tau) = \frac{1}{\sqrt{|\alpha|}}\left[\sqrt{c^2-1}\,\cosh(2\sqrt{|\alpha|}(\tau-\tau_1)) + c\right], \tag{4.16}$$

while for $|c| < 1$

$$g(\tau) = \frac{1}{\sqrt{|\alpha|}}\left[\sqrt{1-c^2}\,\sinh(2\sqrt{|\alpha|}(\tau-\tau_1)) + c\right]. \tag{4.17}$$

Notice that positivity of $f(\sigma)$ restricts the choice of f_1 and f_2 while positivity of $g(\tau)$ in some of these cases restricts the range of τ to a finite or semi-infinite interval.

Let $F(\sigma) = \int d\sigma f(\sigma)$ so that $x(\sigma,\tau) = (F(\sigma) + k(\tau))/\sqrt{g(\tau)}$ for some function $k(\tau)$. We can also show that Z_0/φ_0 is of the form

$$\frac{Z_0}{\varphi_0} = h(\tau) + \frac{\partial_\tau g}{2\sqrt{g}}F\,. \tag{4.18}$$

The functions $h(\tau)$ and $k(\tau)$ can be computed using the equation of motion. Computing $p_\pm$ from (2.5) and (2.8), we get the following relationship

$$\alpha g^2\left(x - \frac{k(\tau)}{\sqrt{g(\tau)}}\right)^2 + \left(p + x\frac{\partial_\tau g(\tau)}{2} + h(\tau) - \frac{k(\tau)\partial_\tau g(\tau)}{2\sqrt{g(\tau)}}\right)^2 = \bar{f}^2 g(\tau) \tag{4.19}$$

which we recognize as an ellipse (a hyperbola) if α is positive (negative). Notice that, from equation (4.14), the compact (elliptical) solutions correspond to a finite range of τ.

The interaction terms (2.12) simplify under our assumption to

$$S_{\rm int} = \int d\tau\, d\sigma\left[\frac{1}{6}\Lambda\,(\partial_\sigma\eta)^3 + \frac{1}{2}\,(\partial_\tau\eta)^2\sum_{n=1}^{\infty}\Lambda^n\,(\partial_\sigma\eta)^n\right], \tag{4.20}$$

where the effective coupling constant is

$$\Lambda = -\frac{\sqrt{\pi}}{\varphi_0}\partial_x\sigma = -\frac{\sqrt{\pi}}{f(\sigma)^2}\,. \tag{4.21}$$

So we find that the coupling constant is time-independent for this class of solutions. We note that the moving-hyperbola solution (3.8) falls into this class.

As long as $|\Lambda\,\partial_\sigma\eta| < 1$, we can sum the series to get

$$S_{\rm int} = \int d\tau\, d\sigma \left[\frac{1}{6}\Lambda\,(\partial_\sigma\eta)^3 + \frac{1}{2}\,(\partial_\tau\eta)^2\left(\frac{\Lambda\partial_\sigma\eta}{1-\Lambda\partial_\sigma\eta}\right)\right]. \tag{4.22}$$

The first interaction term diverges as $\varphi_0 \to 0$, which occurs when the width of the Fermi sea goes to zero. This corresponds to strong coupling at the tip of the static hyperbolic Fermi surface. The second interaction term diverges as $|\Lambda\,\partial_\sigma\eta| \to 1$. We have

$$\Lambda\,\partial_\sigma\eta = -\frac{\sqrt{\pi}}{\varphi_0}\partial_x\eta = \frac{\varphi_0-\varphi}{\varphi_0}\,, \tag{4.23}$$

so the breakdown happens when the excitations become comparable to the width of the Fermi sea (as can also been seen directly from (2.10)). In this case, the Fermi sea may pinch and split into two, so we would not expect to be able to neglect interactions between the upper and lower Fermi surface. Thus, the collective theory becomes strongly coupled exactly in the places one would expect it to from general considerations.

We have demonstrated that, under the restriction (4.1), the action takes a universal, static form (4.20). The natural question to ask is whether such a universal form of the action might exist for all solutions. As a partial answer to this question, in the appendix we analyze explicitly an example which does not fall into the class of solutions studied in this section. We show that, even with the freedom of conformal change of coordinates, it is not always possible to make the interaction term static in Alexandrov coordinates.

5. Fermi droplet cosmology

In this last section, we construct an explicit example of the class of solutions discussed above — a droplet solution in which only a finite region of phase space is filled (so that the Fermi surface is a closed curve). Such solutions are believed to give rise to time dependent backgrounds in the spacetime picture [5], although no precise correspondence has been found so far.

In the simplest case, the Fermi surface is a circle in phase space with radius R and center $(p,x) = (0,x_0)$ at time $t = 0$. Notice that we must demand $x_0 > \sqrt{2}R$ in order for the surface not to cross the diagonals $p = \pm x$ (otherwise, some of the fermions will spill over the potential barrier as the droplet bounces off it).

It is not difficult to write down the evolution of this Fermi surface

$$e^{-2t}(x+p-x_0e^{t})^2 + e^{2t}(x-p-x_0e^{-t})^2 = 2R^2\,. \tag{5.1}$$

Solving for p we find

$$\varphi_0 = \frac{\sqrt{R^2 \cosh 2t - (x - x_0 \cosh t)^2}}{\cosh 2t} \,. \tag{5.2}$$

A sensible σ-coordinate is an angle parameterizing the upper surface, running from 0 to π between the points where $\varphi_0 = 0$. These are given by

$$x = x_0 \cosh t \pm R\sqrt{\cosh 2t}\,, \tag{5.3}$$

so the simplest guess for an Alexandrov coordinate (which we call θ to stress its angular nature) is such that

$$x = x_0 \cosh t - R \cos\theta \sqrt{\cosh 2t}\,. \tag{5.4}$$

Using the second condition in (4.4), we find

$$\partial_t \tau = \frac{1}{\cosh 2t}\,, \tag{5.5}$$

which gives

$$\tau = \tan^{-1}(\tanh t)\,. \tag{5.6}$$

Thus, τ runs over the finite range $-\pi/4 \leq \tau \leq \pi/4$. In these new coordinates, we find

$$x = \frac{1}{\sqrt{\cos 2\tau}}(x_0 \cos\tau - R\cos\theta)\,, \qquad \varphi_0 = R\sqrt{\cos 2\tau}\sin\theta\,. \tag{5.7}$$

It can be checked that these coordinates do fulfill the first condition in (4.4) as well.

We see that

$$\Lambda = -\frac{\sqrt{\pi}}{\varphi_0}\partial_x\theta = -\frac{\sqrt{\pi}}{R^2\sin^2\theta}\,, \tag{5.8}$$

and

$$g(\tau) = \cos 2\tau\,, \qquad f(\theta) = R\sin\theta\,, \tag{5.9}$$

and the action (2.10) simplifies to

$$S = \int d\tau d\theta \left\{ \frac{1}{2}[(\partial_\tau\eta)^2 - (\partial_\theta\eta)^2] - \frac{\sqrt{\pi}}{6R^2\sin^2\theta}(\partial_\theta\eta)^3 + \frac{1}{2}(\partial_\tau\eta)^2 \sum_{n=1}^{\infty}\left(-\frac{\sqrt{\pi}}{R^2\sin^2\theta}\partial_\theta\eta\right)^n \right\}. \tag{5.10}$$

As anticipated, the theory is strongly coupled at the endpoints of the droplet where $\varphi_0 \to 0$. Note that the coordinates are smooth across the steep/shallow divide.[1]

As an aside, consider a modification to the droplet discussed above. At time $t = 0$, replace the regions $\pi/4 < \theta < 3\pi/4$ and $5\pi/4 < \theta < 7\pi/4$ by straight lines so that the droplet takes the form of a rectangle with semi-circular ends. A straightforward computation leads to the conclusion that one can find global coordinates which yield a flat kinetic term in the action. As one might expect, time is still compact as it was in the elliptical case, indicating that the compactness is not merely an accident occurring only for this particular shape.

[1] It is possible to explicitly reach the θ-coordinate from the generally applicable forms (3.2), (3.3) by using appropriate conformal transformations on each patch, but the computation is complicated.

While the droplets are amusing objects in matrix theory, it would be more interesting and satisfying if they had a clear spacetime interpretation in string theory. The collective field description, which we have constructed here, suggests that they have an interpretation as some closed string backgrounds. The massless scalar fluctuations should correspond to some string field, the analog of the tachyon in $c = 1$ Liouville string. The strongly coupled regions at each end of the droplet should correspond to 'tachyon walls' — strongly coupled regions of large tachyon VEV. If such a closed string, worldsheet description could be found, it would provide an example of an open-closed string, finite N duality between a time-dependent finite universe and the matrix quantum mechanics of the $D0$ branes making up the droplet.

Unfortunately, it is not clear how to construct such a spacetime interpretation. The natural time τ is compact, corresponding to the fact that in the fermion time, t, fluctuations of a compact Fermi surface become frozen in the past and future. Also, examining the interaction term, we notice that the coupling Λ is bounded from below so that the theory does not approach a free theory in any region (though the coupling can be made arbitrarily small by taking R large). This makes it unlikely that it will be possible to define an S-matrix. In addition, in the standard $c = 1$ story, the matrix-to-spacetime dictionary is complicated by the presence of leg pole factors, additional phases needed to match the matrix model S-matrix to its string worldsheet counterpart. Supposedly such a complication would appear for the droplet cosmologies as well, but there is no obvious candidate for what it might be. Therefore, it seems unlikely that a spacetime analysis of tachyon scattering can be carried out as has been done in the case of the standard, static Fermi sea as well as the moving hyperbola solution (3.7) [9, 10]. In addition, the leg pole transform is only known on the null boundaries of spacetime, and not in its bulk. This means that there is no dictionary leading to the computation of the tachyon condensate corresponding to the classical fermi sea configuration.

The finite extent of the time τ in Alexandrov coordinates introducies an additional complication, the appearance of boundaries (in this case, spacelike boundaries at $\tau = \pm\pi/4$). What the boundary condition on these should be is not clear. The appearance of boundaries is not unique to compact Fermi surfaces; boundaries of this type, both timelike and spacelike, have appeared in the analysis of noncompact Fermi surfaces in [6]. The question of how these coordinates should be related to spacetime coordinates, or, more broadly, the question of which spacetime metric does the matrix model provide the data in is beyond the scope of this paper.

Perhaps it is possible to find a solution to the effective spacetime theory which would mimic the properties of the droplet cosmology outlined above. This intriguing question is left for future research.

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A. Intrinsically time-dependent actions: an example

In this appendix, we will analyze an example which does not fall into the restricted category of solutions analyzed in section 4. We return to the general case from section 3, and, using only the property (3.1), we write the cubic part of the action as

$$S_{(3)} = \frac{\sqrt{\pi}}{2}\int d\sigma d\tau \frac{1}{6\varphi_0|\partial_x\tau^+\partial_x\tau^-|}\Big\{((\partial_x\tau^-)^3 - (\partial_x\tau^+)^3)((\partial_\sigma\eta)^3 + 3\partial_\sigma\eta(\partial_\tau\eta)^2) - \\ - ((\partial_x\tau^+)^3 + (\partial_x\tau^-)^3)(3(\partial_\sigma\eta)^2\partial_\tau\eta + (\partial_\tau\eta)^3)\Big\}\,. \tag{A.1}$$

For the couplings in this action to be time independent, as in equation (4.20), $\partial_x\tau^\pm/\varphi_0$ must be a function of σ only. We will analyze this condition in a specific example.

Consider the Fermi surface given by

$$x^2 - p^2 = 1 + (x-p)^3 e^{3t}\,. \tag{A.2}$$

Parametrically, this surface is given by

$$\begin{aligned} x &= \cosh\omega + \frac{1}{2}e^{3t-2\omega} \\ p &= \sinh\omega + \frac{1}{2}e^{3t-2\omega}\,. \end{aligned} \tag{A.3}$$

Since the parametric form is similar to the one given in [6], we use the procedure given there to define the Alexandrov coordinates

$$\tau^+ = t - \omega\,, \qquad \tau^- = t - \tilde{\omega}\,, \tag{A.4}$$

where $\tilde{\omega}$ is defined by $x(\tilde{\omega},t) = x(\omega,t)$ as well as $p(\omega,t) = p_+$ and $p(\tilde{\omega},t) = p_-$. It is possible to solve for x, t and $p_\pm$ as functions of $\tau^\pm$:

$$\begin{aligned} x(\tau^\pm) &= -\frac{e^{2\tau^+ + 2\tau^-} - e^{\tau^+} - e^{\tau^-}}{2\sqrt{e^{\tau^+ + \tau^-} - e^{2\tau^+ + 3\tau^-} - e^{3\tau^+ + 2\tau^-}}} \\ \exp(t(\tau^\pm)) &= -\frac{\sqrt{e^{\tau^+ + \tau^-} - e^{2\tau^+ + 3\tau^-} - e^{3\tau^+ + 2\tau^-}}}{e^{2\tau^+ + \tau^-} + e^{\tau^+ + 2\tau^-} - 1} \\ p_+(\tau^\pm) &= \frac{e^{2\tau^+ + 2\tau^-} + 2e^{3\tau^+ + \tau^-} + e^{\tau^-} - e^{\tau^+}}{2\sqrt{e^{\tau^+ + \tau^-} - e^{2\tau^+ + 3\tau^-} - e^{3\tau^+ + 2\tau^-}}} \\ p_-(\tau^\pm) &= \frac{e^{2\tau^+ + 2\tau^-} + 2e^{3\tau^- + \tau^+} + e^{\tau^+} - e^{\tau^-}}{2\sqrt{e^{\tau^+ + \tau^-} - e^{2\tau^+ + 3\tau^-} - e^{3\tau^+ + 2\tau^-}}}\,. \end{aligned} \tag{A.5}$$

The coordinates given here have the property that the edge of the Fermi sea ($p_+ = p_-$) is at $2\sigma = \tau^+ - \tau^- = 0$. It is now possible to compute $\partial_x\tau^\pm/\varphi_0$. Not surprisingly, this is not a function of σ only. The question is whether, by a suitable conformal change of coordinates to $\bar{\tau}^\pm$, this condition could be satisfied. The change of coordinates would have to map $\sigma = 0$ to itself to maintain a static Fermi sea edge in the new coordinates. Thus, the change of coordinates must be of the form $\tau^{\pm} = f(\bar{\tau}^{\pm})$, with $f(\cdot)$ an arbitrary function. Define $Q_{\pm} \equiv \partial_x \tau^{\pm}/\varphi_0$. The necessary condition is then

$$0 = \partial_{\bar{\tau}} Q_{\pm} = f'(\bar{\tau}^{+})\partial_{\tau^{+}} Q_{\pm} + f'(\bar{\tau}^{-})\partial_{\tau^{-}} Q_{\pm} \tag{A.6}$$

implying that $\partial_{\tau^{-}} Q_{\pm}/\partial_{\tau^{+}} Q_{\pm}$ is of the form

$$W_{\pm}(\tau^{+}, \tau^{-}) \equiv \frac{\partial_{\tau^{-}} Q_{\pm}}{\partial_{\tau^{+}} Q_{\pm}} = -\frac{f'(\bar{\tau}^{+})}{f'(\bar{\tau}^{-})} = -\frac{F(\tau^{+})}{F(\tau^{-})}\,. \tag{A.7}$$

Therefore,

$$W_{\pm}(\tau^{+}, \tau^{-}) W_{\pm}(\tau^{-}, \tau^{+}) = 1\,. \tag{A.8}$$

By explicit computation, it can be checked that this condition is not satisfied. Therefore, there does not exist a coordinate transformation after which $S_{(3)}$ has no τ dependence.

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